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Resonant wave interactions in a stratified shear flow

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Resonant interactions between triads of internal gravity waves propagating in a shear flow are considered for the case when the stratification and the background shear flow vary slowly with respect to typical wavelengths. If $\omega_n, \boldsymbol{\kappa}_n (n = 1,2,3)$ are the local frequencies and wavenumbers respectively then the resonance conditions are that $\omega_1+\omega_2+\omega_3 = 0$ and $\boldsymbol{\kappa}_1+\boldsymbol{\kappa}_2+\boldsymbol{\kappa}_3 = 0$. If the medium is only weakly inhomogeneous, then there is a strong resonance and to leading order the resonance conditions are satisfied globally. The equations governing the wave amplitudes are then well known, and have been extensively discussed in the literature. However, if the medium is strongly inhomogeneous, then there is a weak resonance and the resonance conditions can only be satisfied locally on certain space–time resonance surfaces. The equations governing the wave amplitudes in this case are derived, and discussed briefly. Then the results are applied to a study of the hierarchy of wave interactions which can occur near a critical level, with the aim of determining to what extent a critical layer can reflect wave energy.

1. Introduction

Resonant interactions between wave triads have been extensively studied over the last twenty years particularly in the areas of nonlinear optics, plasma physics and fluid dynamics. See, for instance, the texts by Bloembergen (1965), Davidson (1972), Weiland & Wilhelmsson (1977) and Craik (1986). For the most part, attention has focused on the case when the resonance conditions are met globally. In the case when the interacting waves are infinite periodic wavetrains the interaction equations can be integrated in terms of elliptic functions, and the nature of the interaction is well understood. If, instead, the interacting waves are wave packets, the interaction equations can be solved by the inverse scattering transform algorithm and many interesting solutions have been found (see, for instance, Zakharov & Manakov 1975; Bers, Kaup & Reiman 1976; Craik 1978; Kaup, Reiman & Bers 1979).

In this paper we explore a situation when the resonance conditions can only be met locally on certain space–time surfaces. The context is the interaction between internal gravity waves propagating in a stratified shear flow, for the case when the stratification and background shear flow vary slowly on lengthscales and timescales associated with the waves. Resonant interactions between internal gravity waves in the absence of any background shear flow have been studied by Ball (1964) for a two-layer fluid and Thorpe (1966) for continuous stratification. Resonant interactions between interfacial waves in layered fluids have been discussed by Cairns (1979), Craik & Adams (1979) and Tsutahara (1984, 1986). They showed that explosive interactions were possible when there were appropriate velocity jumps between the layers. The essential difference between these studies, and the case discussed here, is that in all these studies the resonance conditions are met globally, whereas here the resonance conditions are only satisfied on certain space–time surfaces.

One of the main motivations for our study is the series of papers by Brown & Stewartson (1980, 1982*a*, *b*) on the nonlinear processes affecting internal gravity waves near a critical level. They showed that wave reflection and transmission at a critical level was essentially determined by a hierarchy of wave interactions, although their study differs in a significant way from that described here in that higher harmonics of the main incoming wave play a crucial role in initiating the interaction process. In contrast, here only the first harmonics of the waves (i.e. that part which satisfies the linearized equations of motion) participate in the interaction. This aspect of our study will be developed further in §5. In §2 the derivation of the interaction equations is described, and in §3 we present a general analysis of these equations. Then in §4 we describe in detail the special case of wave interactions near a critical level.

2. Derivation of the interaction equations

Let the basic flow consist of the horizontal shear flow $\boldsymbol{u}_0(z) = (u_0(z), v_0(z), 0)$ and density field $\rho_0(z)$. Here z is the vertical coordinate. Throughout we shall use non-dimensional variables based on a lengthscale h_1 (a typical wavelength), a timescale N_1^{-1} (where N_1 is a typical value of the Brunt–Väisälä frequency) and a pressure scale $\rho_1 g h_1$ (where ρ_1 is a typical value of the density). The density gradient is given by

$$\frac{\mathrm{d}\rho_0}{\mathrm{d}z} = -\beta\rho_0 N^2, \tag{2.1}$$

where $N(z)$ is the non-dimensional Brunt–Väisälä frequency and $\beta = h_1 N_1^2 g^{-1}$, which is small in the Boussinesq approximation. Relative to this basic flow we define the particle displacement $\boldsymbol{\xi}(\boldsymbol{x}, t)$ so that the Eulerian coordinate $\hat{\boldsymbol{x}}$ of a fluid particle is

$$\hat{\boldsymbol{x}} = \boldsymbol{x} + \boldsymbol{\xi}. \tag{2.2}$$

Here $\boldsymbol{x}$ is a Lagrangian coordinate convected with the basic flow. The Lagrangian equations of motion in the present context have been described by Grimshaw (1981). First, in a non-diffusive incompressible fluid the density is a material property and is again given by $\rho_0(z)$. Next the Jacobian of the transformation from $\boldsymbol{x}$ to $\hat{\boldsymbol{x}}$ is constant, and equal to one. It follows that

$$\nabla\cdot\boldsymbol{\xi} + I = 0, \tag{2.3a}$$

where

$$I = \tfrac{1}{2}\nabla\cdot\{\boldsymbol{\xi}(\nabla\cdot\boldsymbol{\xi}) - (\boldsymbol{\xi}\cdot\nabla)\,\boldsymbol{\xi}\} + \det\left[\frac{\partial\boldsymbol{\xi}}{\partial\boldsymbol{x}}\right]. \tag{2.3b}$$

If p is the pressure, we define a pressure perturbation $q(\boldsymbol{x}, t)$ by

$$p = p_0(z) - \rho_0(z)\,\zeta + \beta q, \tag{2.4}$$

where $p_0(z)$ is the basic pressure field, and ζ is the vertical particle displacement. Then the momentum equation is

$$\rho_0\frac{\mathrm{d}^2\boldsymbol{\xi}}{\mathrm{d}t^2} + \rho_0 N^2\zeta\boldsymbol{k} + \nabla q + \boldsymbol{M} = \boldsymbol{D}, \tag{2.5a}$$

where
$$\boldsymbol{M} = \rho_0 \frac{d^2\boldsymbol{\xi}}{dt^2} \cdot \frac{\partial \boldsymbol{\xi}}{\partial \boldsymbol{x}}, \tag{2.5b}$$

and
$$\frac{d}{dt} = \frac{\partial}{\partial t} + \boldsymbol{u}_0 \cdot \boldsymbol{\nabla}. \tag{2.5c}$$

Here $\boldsymbol{D}$ is a term representing dissipative effects. Eliminating q and $\boldsymbol{\xi}_H$ (the horizontal part of $\boldsymbol{\xi}$) from the linear parts of equations (2.3a) and (2.5a) we find that
$$\frac{\partial}{\partial z}\left(\rho_0 \frac{d^2}{dt^2}\left(\frac{\partial \zeta}{\partial z}\right)\right) + \rho_0\left(\frac{d^2}{dt^2} + N^2\right)\nabla_H^2 \zeta + M = D, \tag{2.6a}$$

where
$$M = \nabla_H^2 M_V - \boldsymbol{\nabla}_H \cdot \frac{\partial \boldsymbol{M}_H}{\partial z} + \frac{\partial}{\partial z}\left(\rho_0 \frac{d^2 I}{dt^2}\right), \tag{2.6b}$$

and
$$D = \nabla_H^2 D_V - \boldsymbol{\nabla}_H \cdot \frac{\partial \boldsymbol{D}_H}{\partial z}. \tag{2.6c}$$

Here the subscripts 'H' and 'V' denote horizontal and vertical components respectively. In (2.6a) M contains all the nonlinear terms, and D contains all the dissipative terms.

Next we assume that the basic flow varies slowly with respect to the lengthscales and timescales associated with the wave field. Hence we introduce the slow variables
$$\boldsymbol{X} = \epsilon \boldsymbol{x}, \quad T = \epsilon t, \tag{2.7}$$
where ϵ is a small parameter. The basic flow is assumed to be a function of $Z = \epsilon z$, so that $\boldsymbol{u}_0 = \boldsymbol{u}_0(Z)$ and $\rho_0 = \rho_0(Z)$. Consistent with these hypotheses we assume that the Boussinesq parameter β is $O(\epsilon)$, and we put $\beta = \sigma\epsilon$. Then (2.1) becomes
$$\frac{D\rho_0}{DZ} = -\sigma \rho_0 N^2. \tag{2.8}$$

We shall also assume that the dissipative terms are $O(\epsilon)$ and replace D with ϵD. Next we write (2.6a) in the form
$$L\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial \boldsymbol{x}}; z; \epsilon\right)\zeta + M = \epsilon D, \tag{2.9}$$

Here $L(p_0, \boldsymbol{p}; z; \epsilon)$ is a linear operator defined by
$$L = L_0 + \epsilon L_1, \tag{2.10a}$$

where
$$L_0(p_0, \boldsymbol{p}; z) = \rho_0\{(p_0 + \boldsymbol{u}_0 \cdot \boldsymbol{p}_H)^2 |\boldsymbol{p}|^2 + N^2 |p_H|^2\}, \tag{2.10b}$$

and
$$L_1(p_0, \boldsymbol{p}; z) = \frac{1}{2}\frac{\partial}{\partial z}\left(\frac{\partial L_0}{\partial p_3}\right). \tag{2.10c}$$

Here $p_3 = \boldsymbol{p} \cdot \boldsymbol{k}$ is the vertical component of $\boldsymbol{p}$. We shall now seek an asymptotic solution which describes a set of interacting modulated waves. Thus we put
$$\zeta = \alpha\left\{\sum_r A_r(\boldsymbol{X}, T) \exp(i\theta_r) + *\right\} + \alpha^2 \zeta^{(1)} + \ldots, \tag{2.11a}$$

where
$$\theta_r = \frac{1}{\epsilon}\Theta_r(\boldsymbol{X}, T). \tag{2.11b}$$

Each wave is described by a slowly varying amplitude A_r and a rapidly varying phase θ_r. For each wave the local frequency ω_r, and the local wavenumber, $\boldsymbol{\kappa}_r$, are defined by

$$\omega_r = -\frac{\partial \Theta_r}{\partial T}, \quad \boldsymbol{\kappa}_r = \nabla \Theta_r. \tag{2.12}$$

The required balance between slow modulations and nonlinearity which determines the relation between ϵ and α will be considered below.

As a preliminary to obtaining the solution we define the dispersion operator

$$\mathscr{D}(\omega, \boldsymbol{\kappa}; Z) \equiv L_0(-\mathrm{i}\omega, \mathrm{i}\boldsymbol{\kappa}; Z), \tag{2.13a}$$

or

$$\mathscr{D} \equiv \rho_0\{\hat{\omega}^2\kappa^2 - N^2\kappa_{\mathrm{H}}^2\}, \tag{2.13b}$$

where

$$\hat{\omega} \equiv \omega - \boldsymbol{u}_0 \cdot \boldsymbol{\kappa}_{\mathrm{H}}, \quad \kappa = |\boldsymbol{\kappa}|, \quad \kappa_{\mathrm{H}} = |\boldsymbol{\kappa}_{\mathrm{H}}|. \tag{2.13c}$$

Here (2.13*b*) follows from (2.10*b*). Substituting (2.11*a*) into (2.9), we find that, at leading order,

$$\mathscr{D}(\omega_r, \boldsymbol{\kappa}; Z) = 0. \tag{2.14}$$

Thus, not unexpectedly, each wave satisfies the well-known dispersion relation for internal gravity waves. At the next order we obtain

$$\begin{aligned}\alpha^2 L_0 \zeta^{(1)} + \alpha\epsilon \Bigg\{ \sum_r \Bigg[\frac{\partial L_0}{\partial p_0}\frac{\partial A_r}{\partial T} + \frac{\partial L_0}{\partial \boldsymbol{p}} \cdot \nabla A_r + \frac{1}{2}\frac{\partial}{\partial T}\left(\frac{\partial L_0}{\partial p_0}\right) A_r \\ + \tfrac{1}{2}\nabla \cdot \left(\frac{\partial L_0}{\partial \boldsymbol{p}}\right) A_r + L_1 A_r - \frac{1}{2}\left(\frac{\partial}{\partial Z}\right)_e \left(\frac{\partial L_0}{\partial p_3}\right) A_r \Bigg] \exp(\mathrm{i}\theta_r) + * \Bigg\} \\ + M_2 + O(\epsilon^2\alpha, \epsilon\alpha^2, \alpha^3) = \epsilon D_1.\end{aligned} \tag{2.15}$$

Here M_2 is the 'α^2' term in the nonlinear expression M, and D_1 is the 'α' term in the dissipative expression D. The balance between modulation and nonlinearity requires that $\epsilon = O(\alpha)$. Also, in the second term $\partial L_0/\partial p_0$ and $\partial L_0/\partial \boldsymbol{p}$ are evaluated at $p_0 = -\mathrm{i}\omega_r, \boldsymbol{p} = \mathrm{i}\boldsymbol{\kappa}_r$. To avoid secularities it is clear that $\zeta^{(1)}$ cannot contain any terms whose phase is θ_r, since L_0 is then a null operator. It follows that, to leading order,

$$\begin{aligned}\mathrm{i}\alpha\epsilon \left\{ \frac{\partial \mathscr{D}}{\partial \omega_r}\frac{\partial A_r}{\partial T} - \frac{\partial \mathscr{D}}{\partial \boldsymbol{\kappa}_r} \cdot \nabla A_r + \frac{1}{2}\left[\frac{\partial}{\partial T}\left(\frac{\partial \mathscr{D}}{\partial \omega_r}\right) - \nabla \cdot \left(\frac{\partial \mathscr{D}}{\partial \boldsymbol{\kappa}_r}\right)\right] A_r \right\} \\ + M_2^{(r)} \exp(-\mathrm{i}\theta_r) = \epsilon D_1^{(r)} \exp(-\mathrm{i}\theta_r).\end{aligned} \tag{2.16}$$

Here we have used (2.10*c*) and (2.13*a*), and the superscript 'r' in M_2 and D_1 denotes those terms whose phase is approximately θ_r. Finally using (2.14) we find that (2.16) becomes

$$\frac{\mathrm{i}\alpha\epsilon}{2A_r}\left\{\frac{\partial}{\partial T}\left(\frac{\partial \mathscr{D}}{\partial \omega_r}A_r^2\right) + \nabla \cdot \left(\boldsymbol{V}_r \frac{\partial \mathscr{D}}{\partial \omega_r}A_r^2\right)\right\} + M_2^{(r)} \exp(-\mathrm{i}\theta_r) = \epsilon D_1^{(r)} \exp(-\mathrm{i}\theta_r), \tag{2.17a}$$

where

$$\boldsymbol{V}_r = \frac{\partial \omega_r}{\partial \boldsymbol{\kappa}_r}. \tag{2.17b}$$

Here $\boldsymbol{V}_r$ is the group velocity, and in the absence of the nonlinear and dissipative terms, equation (2.17*a*) describes conservation of wave action (see, for instance, Grimshaw 1974). We note that

$$\mathscr{A}_r = \frac{\partial \mathscr{D}}{\partial \omega_r}|A_r|^2 = 2\rho_0 \hat{\omega}_r \kappa_r^2 |A_r|^2 \tag{2.18}$$

is the wave action density in the present context. Note that in the absence of the nonlinear and dissipative terms, the conserved quantity in (2.17*a*) is the complex wave action density $\mathscr{A}_r \exp(2\mathrm{i} \arg A_r)$, and consequently both $\mathscr{A}_r$ and $\arg(A_r)$ satisfy conservation equations. For the case when the dissipative term is due to molecular viscosity we find that (see Grimshaw 1974)

$$\epsilon D_1^{(r)} \exp(-\mathrm{i}\theta_r) = -\tfrac{1}{2}\mathrm{i}\alpha\epsilon\nu\kappa_r^2 \frac{\partial \mathscr{D}}{\partial \omega_r} A_r, \tag{2.19}$$

where ν is the kinematic viscosity.

We now turn to an examination of the nonlinear term in (2.17*a*). From the discussion above we see that this term contributes only in the vicinity of a resonance, where a pair of waves, with phases θ_p and θ_q, satisfy a resonance condition. This is defined as follows. Let

$$\theta_p + \theta_q + \theta_r = \frac{1}{\epsilon}\chi(\boldsymbol{X}, T), \tag{2.20a}$$

so that

$$\omega_p + \omega_q + \omega_r = -\frac{\partial \chi}{\partial T}, \tag{2.20b}$$

and

$$\boldsymbol{\kappa}_p + \boldsymbol{\kappa}_q + \boldsymbol{\kappa}_r = \boldsymbol{\nabla}\chi. \tag{2.20c}$$

Then a resonance occurs when $\partial\chi/\partial T$ and $\boldsymbol{\nabla}\chi$ vanish simultaneously. In general, these conditions can only be met on isolated manifolds in $(\boldsymbol{X}, T)$-space. The contribution of the p- and q-waves to the nonlinear term in (2.17*a*) can now be evaluated from (2.3*b*), (2.5*b*) and (2.6*b*). Omitting details we find that

$$M_2^{(r)} \exp(-\mathrm{i}\theta_r) = -\mathrm{i}\alpha^2\gamma A_p^* A_q^* \exp(-\mathrm{i}\chi/\epsilon), \tag{2.21a}$$

where

$$\gamma = \rho_0 m_r \hat{\omega}_r^2 (\boldsymbol{\eta}_q \cdot \boldsymbol{\kappa}_r)(\boldsymbol{\eta}_p \cdot \boldsymbol{\kappa}_r) + \ldots + \ldots, \tag{2.21b}$$

and

$$\boldsymbol{\eta}_r = \boldsymbol{k} - m_r \frac{\boldsymbol{\kappa}_{Hr}}{\kappa_{Hr}^2}, \quad m_r = \boldsymbol{\kappa}_r \cdot \boldsymbol{k}. \tag{2.21c}$$

The omitted terms in (2.21*b*) are obtained by cyclic interchange of the indices p, q, r in the displayed term. We note that $\{A_r \boldsymbol{\eta}_r \exp(\mathrm{i}\theta_r) + *\}$ is the particle displacement for the r-wave. In evaluating the coefficient γ we use the fact that the nonlinear term (2.21*a*) is only significant near resonance and hence we can use the resonance conditions (i.e. (2.20*a*–*c*) with the right-hand side replaced by zero). The result that γ is then real-valued and independent of the ordering of the indices p, q, r is a consequence of the fact that the non-dissipative part (i.e. the left-hand sides) of equations (2.3*a*) and (2.5*a*) can be derived from a Lagrangian (cf. Hasselman 1966). Combining (2.17*a*), (2.18) and (2.21*a*) we obtain the interaction equations in the form,

$$\frac{1}{2A_r}\left\{\frac{\partial}{\partial T}\left(\frac{\partial \mathscr{D}}{\partial \omega_r} A_r^2\right) + \boldsymbol{\nabla}\cdot\left(\boldsymbol{V}_r \frac{\partial \mathscr{D}}{\partial \omega_r} A_r^2\right) + \nu\kappa_r^2 \frac{\partial \mathscr{D}}{\partial \omega_r} A_r^2\right\} = \frac{\alpha}{\epsilon}\gamma A_p^* A_q^* \exp\left(-\frac{\mathrm{i}\chi}{\epsilon}\right). \tag{2.22}$$

There are two similar equations for the p- and q-waves which are obtained by a cyclic rotation of the indices p, q, r. Further discussion of these equations together with determination of the possible ordering relationships between the parameters α and ϵ will be taken up in the next section.

3. Analysis of the interaction equations

In the analysis of the interaction equations (2.22) two main cases can be distinguished, depending on whether the resonance conditions can be satisfied globally, or locally. We shall call these two cases (i) and (ii) respectively.

(i) *Weak inhomogeneity, or strong resonance.* In this case the inhomogeneous terms in the dispersion relation (2.14) are $O(\epsilon)$ (i.e. N^2 and $\boldsymbol{u}_0$ are constant to $O(\epsilon)$), with the consequence that ω_r and $\boldsymbol{\kappa}_r$ are constants to $O(\epsilon)$. We may write

$$\omega_r = \omega_r^{(0)} + \epsilon\omega_r^{(1)}(\boldsymbol{X},T), \quad \boldsymbol{\kappa}_r = \boldsymbol{\kappa}_r^{(0)} + \epsilon\boldsymbol{\kappa}_r^{(1)}(\boldsymbol{X},T), \tag{3.1}$$

where $\omega_r^{(0)}$ and $\kappa_r^{(0)}$ are constants, and satisfy the resonance conditions,

$$\omega_p^{(0)} + \omega_q^{(0)} + \omega_r^{(0)} = 0, \quad \boldsymbol{\kappa}_r^{(0)} + \boldsymbol{\kappa}_q^{(0)} + \boldsymbol{\kappa}_q^{(0)} = 0. \tag{3.2a, b}$$

Thus the resonance conditions are met, to $O(\epsilon)$, over the whole $(\boldsymbol{X},T)$-space. We may then put $\chi = \epsilon\chi^{(1)}$ (see (2.20a–c)), and the interaction equations (2.22) become

$$\left(\frac{\partial\mathscr{D}}{\partial\omega_r}\right)^{(0)}\left\{\frac{\partial A_r}{\partial T} + \boldsymbol{V}_r^{(0)}\cdot\boldsymbol{\nabla}A_r + \nu\kappa_r^{(0)2}A_r\right\} = \frac{\alpha}{\epsilon}\gamma^{(0)}A_p^*A_q^*\exp(-\mathrm{i}\chi^{(1)}). \tag{3.3}$$

Clearly the balance between nonlinearity and modulation requires that $\alpha = \epsilon$. Here the superscript '0' indicates quantities evaluated at the lowest order from the expansions (3.1), and such quantities are clearly constants. $\chi^{(1)}$ represents an inhomogeneous detuning term. When $\chi^{(1)}$ is a constant, which may be equated with zero without loss of generality, and the dissipative term is absent (i.e. $\nu = 0$), then it is well known that the interaction equations (3.3) can be solved exactly by the inverse-scattering technique, with a host of interesting exact solutions (see, for instance, Zakharov & Manakov (1975), Bers *et al.* (1976), Craik (1978) and Kaup *et al.* 1979). When $\chi^{(1)}$ is at most a quadratic in $\boldsymbol{X}$ and T, then Reiman, Bers & Kaup (1977) and Reiman (1979) have shown that, in general, there exist transformations involving the phase of A_r, which reduce (3.3) to the case when $\chi^{(1)}$ is zero. We shall not discuss this case any further here as the solutions are well documented in the literature, and it is not the situation we wish to discuss in this paper.

(ii) *Strong inhomogeneity, or weak resonance.* In this case the inhomogeneous terms in the dispersion relation (2.14) are $O(1)$ (i.e. N^2 and $\boldsymbol{u}_0$ are non-trivial functions of Z), with the consequence that ω_r and $\boldsymbol{\kappa}_r$ are non-trivial functions of $\boldsymbol{X}$ and T. The resonance conditions are

$$\omega_p + \omega_q + \omega_r = 0, \quad \boldsymbol{\kappa}_p + \boldsymbol{\kappa}_q + \boldsymbol{\kappa}_r = 0, \tag{3.4a, b}$$

which define a manifold in $(\boldsymbol{X},T)$-space. Here we shall consider only the case of most interest to us when this manifold is a surface, which we shall call the resonance surface, $R(\boldsymbol{X},T) = 0$. The cases when the manifold has a lower dimension will not be considered here, and are left to the reader. On the resonance surface χ is a constant which we shall set to zero. The resonance conditions (3.4a, b) then imply that $\boldsymbol{\nabla}\chi$ and χ_T vanish on the resonance surface $R = 0$ (see (2.20b, c)). It follows that near the resonance surface $\chi \propto R^2$ and the phase in the nonlinear term in (2.22) is significant only when R is $O(\epsilon^{\frac{1}{2}})$. Hence we re-scale, and put

$$R = \epsilon^{\frac{1}{2}}\tau, \quad \chi = \epsilon\phi. \tag{3.5}$$

Here τ is a coordinate transverse to the resonance surface. The remaining coordinates lie in the resonance surface, scale with $\boldsymbol{X}, T$, and, to leading order, are passive during the resonant interaction. The balance between nonlinearity and modulation requires that $\alpha = \epsilon^{\frac{1}{2}}$. Changing variables as indicated, and using (3.5) we find that (2.22) becomes, omitting error terms which are relatively $O(\epsilon^{\frac{1}{2}})$,

$$\delta_r \frac{\partial \mathscr{D}}{\partial \omega_r} \frac{\partial A_r}{\partial \tau} = \gamma A_p^* A_q^* \exp(-\mathrm{i}\phi), \tag{3.6a}$$

where

$$\delta_r = \frac{\partial R}{\partial T} + \boldsymbol{V}_r \cdot \boldsymbol{\nabla} R. \tag{3.6b}$$

The three equations (3.6*a*) are the required interaction equations, in which we may assume that $\phi = \frac{1}{2}S\tau^2$. The coefficients $\delta, \partial\mathscr{D}/\partial\omega_r, \gamma$ and S are evaluated as constants as far as the interaction equations (3.6*a*) are concerned. With suitable changes of scale for the amplitudes A_r, the interaction equations (3.6*a*) may be put into the canonical form

$$\beta_r \frac{\partial A_r}{\partial \tau} = A_p^* A_q^* \exp(-\tfrac{1}{2}\mathrm{i}S\tau^2), \tag{3.7a}$$

where

$$\beta_r = \operatorname{sign}\left(\delta_r \frac{\partial \mathscr{D}}{\partial \omega_r}\right). \tag{3.7b}$$

The sign of the coefficient δ_r determines which side of the resonance surface the r-wave is approaching, while the sign of $\partial\mathscr{D}/\partial\omega_r$ is sign $(\hat{\omega}_r)$ (see (2.13*b*), which is also the sign of the wave action density $\mathscr{A}_r$ (see (2.18)). The timescale for the interaction is $O(\epsilon^{\frac{1}{2}})$ on the T-scale, and during this time χ is $O(\epsilon)$ (see (3.5)), so that during the interaction the right-hand sides of (2.20*b, c*) are $O(\epsilon)$. We also note that the dissipative term in (2.22) is relatively $O(\epsilon^{\frac{1}{2}})$ during the interaction, and hence does not appear in (3.6*a*).

Somewhat surprisingly, equation (3.7*a*) appears not to be exactly integrable, even though the inhomogeneous detuning term in (3.7*a*) is quadratic in τ. The phase-dependent transformation which reduces (3.3) to an integrable equation when $\chi^{(1)}$ is a quadratic function of $\boldsymbol{X}$ and T and $\nu = 0$, depends on the fact that the left-hand side conserves functions of $(\boldsymbol{X} - \boldsymbol{V}_r^{(0)} T)$. Owing to the local scaling (3.5) no such phase-transformation is available for (3.7*a*). Further, since here $\boldsymbol{V}_r$ in (2.22) is a function of $(\boldsymbol{X}, T)$, and χ in (2.22) is not generally a quadratic function of $(\boldsymbol{X}, T)$ (although it is locally near the resonance surface), no such phase-transformation is available for (2.22) either. Since (3.7*a*) can apparently not be solved exactly, we must resort to numerical procedures and analytical approximations. These have been extensively discussed by Grimshaw (1987). Here we shall give only a brief summary of the main results. First we note the conservation laws,

$$\beta_p |A_p|^2 + C_p = \beta_q |A_q|^2 + C_q = \beta_r |A_r|^2, \tag{3.8}$$

where $C_{p,q}$ are constants. These equations are a consequence of wave action conservation in a direction normal to the resonance surface. Indeed the wave action flux normal to the resonance surface for the r-wave is proportional to $\delta_r \partial\mathscr{D}/\partial\omega_r |A_r|^2$ in terms of the unscaled variables used in equation (3.6*a*), or simply $\beta_r |A_r|^2$ for the scaled variables used in equation (3.7*a*). Similarly it may be shown that the wave energy flux normal to the resonance surface is $\hat{\omega}_r \beta_r |A_r|^2$ and the pseudo-energy flux is $\omega_r \beta_r |A_r|^2$, in terms of the scaled variables used in equation (3.7*a*). For general definitions of wave energy and pseudo-energy see Grimshaw (1984). The conservation laws (3.8) together with the resonance conditions (3.4*a, b*) show that the total wave energy flux and the total pseudo-energy flux for all three waves are conserved during the interaction. Note that (2.13c) together with the resonance conditions (3.4a, b) imply that

$$\hat{\omega}_p + \hat{\omega}_q + \hat{\omega}_r = 0. \tag{3.9}$$

It follows that wave action densities $\mathscr{A}_r$, which have the same sign as $\hat{\omega}_r$ (see (2.18)), must have opposite signs during the interaction (i.e. either two have positive values and one a negative value, or vice versa). However the wave energy densities are $\hat{\omega}\mathscr{A}_r$, and hence are always positive. The pseudo-energy densities are $\omega_r \mathscr{A}_r$ and may in general have any combination of signs.

There is no loss of generality in supposing that $\partial R/\partial T > 0$ so that τ (3.5) increases as the resonance surface is traversed. Away from the resonance surface each wave propagates independently and so the boundary conditions for (3.7a) are the specification of $|A_r|^2$ as $\tau \rightarrow -\infty$ and the aim is to determine $|A_r|^2$ as $\tau \rightarrow +\infty$. The nature of the solutions depends on the signs of the coefficients β_r (3.7b) (Grimshaw 1987). First, if all the coefficients $\beta_n\,(n = p, q, r)$ have the same sign, there is a potentiality for growth in all three waves as the resonance surface is traversed. Indeed the solutions obtained by Grimshaw (1987) show that this is generally the case, and that if the parameter S is sufficiently small, there may be an explosive interaction in which an algebraic singularity develops. In the literature explosive interactions have been identified with the presence of both positive-energy waves and negative-energy waves (Bloembergen 1965; Davidson 1972; Weiland & Wilhelmsson 1977), where energy refers to the total energy associated with the wave, and here can be interpreted as the pseudo-energy (see Grimshaw 1984). However, the explosive interaction can only be realized if it is consistent with a well-posed initial condition at $T = 0$ say. That is, the initial condition must be free of singularities, including any possible singularities near $R = 0$ due to an explosive interaction. Note that the resonance surface $R = 0$ may intersect the initial surface $T = 0$. To show that explosive interactions cannot be realized in the present circumstances, we reconsider (2.22), and recall that (3.7a) is a local approximation to (2.22). It is a consequence of (2.22) that

$$\frac{\partial \mathscr{A}_r}{\partial T} + \nabla \cdot (V_r \mathscr{A}_r) + \nu k_r^2 \mathscr{A}_r \tag{3.10}$$

is equal to the same expression with index p or q, where $\mathscr{A}_r$ (2.18) is $\partial \mathscr{D}/\partial \omega_r |A_r|^2$, the wave action density. This result implies that the analogue of (3.8) with respect to the time variable T is that in the absence of dissipation (i.e. $\nu = 0$),

$$\int \mathscr{A}_p \,\mathrm{d}X + C_p = \int \mathscr{A}_q \,\mathrm{d}X + C_q = \int \mathscr{A}_r \,\mathrm{d}X, \tag{3.11}$$

where $C_{p,q}$ are again constants, and the integrals are taken over wave packets, assumed to be locally confined. Since the wave action densities must have opposite signs (a consequence of (3.9)) it follows that with respect to the time variable the interaction is contained. That is, if the amplitudes $|A_r|^2$ are bounded initially, then it is a consequence of (3.11) that they remain bounded. The fact that (3.7a) may have singular solutions describing an explosive interaction is not relevant here, as the above argument demonstrates that sensible initial conditions will lead to boundary conditions for (3.7a) which exclude the growth of a singularity. This result is in marked contrast to the analyses of Cairns (1979) and Craik & Adam (1979) who found explosive interactions for interfacial waves for layered fluids when there were appropriate velocity shears between the layers. The difference here is that although

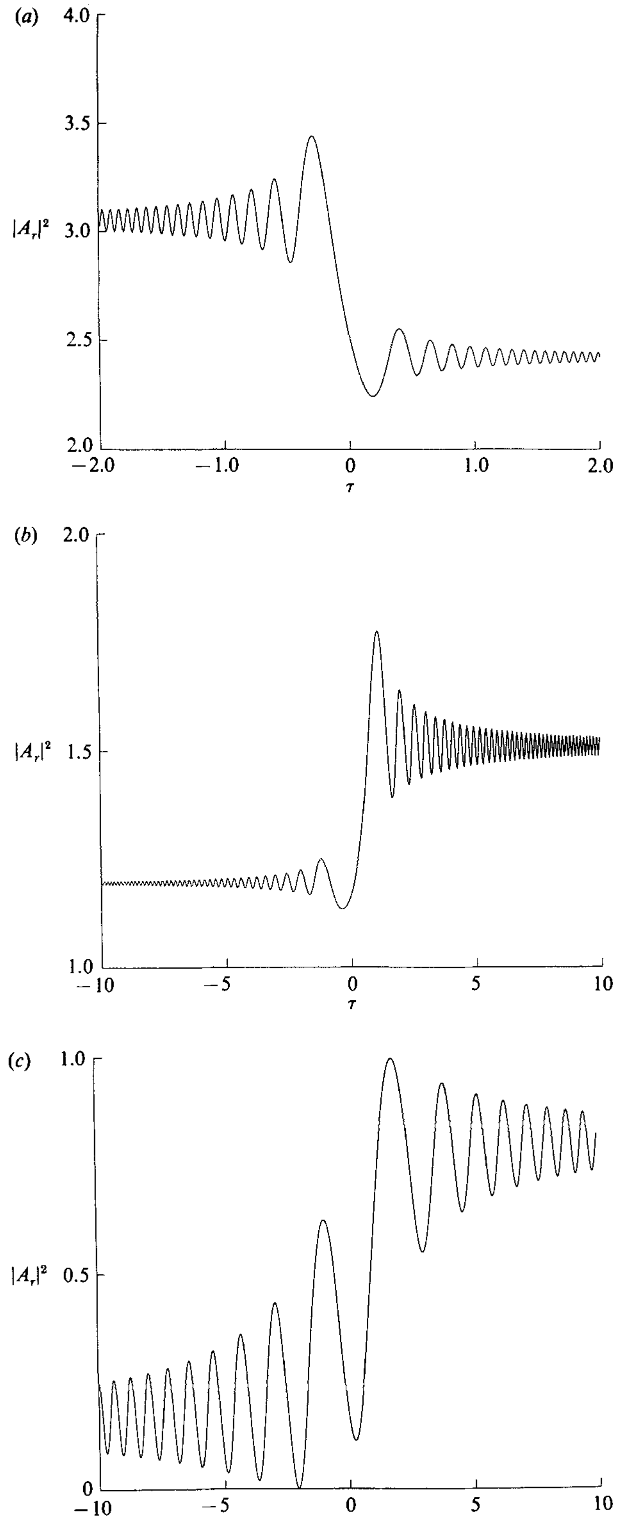

FIGURE 1. A plot of $|A_r|^2$ as a function of τ, determined from numerical solutions of (3.7*a*). (*a*) $S = 50, \beta_p = \beta_q = \beta_r = 1$; (*b*) $S = 5, \beta_p = \beta_q = \beta_r = 1$; (*c*) $S = 1, \beta_p = \beta_q = -\beta_r = 1$.

the waves are propagating in a stratified shear flow, the local dispersion relation (2.14) does not depend explicitly on the velocity shear: the basic flow $\boldsymbol{u}_0(Z)$ enters only through the Doppler-shifted frequency $\hat{\omega}$ (see (2.13*c*)) with the consequence that the resonance conditions (3.4*a*, *b*) imply that (3.9) holds, and the wave action densities have opposite signs. However, in the analyses of Cairns (1979) and Craik & Adam (1979) the local dispersion relation depends directly on the velocity shear, and the analogue of (3.9) does not hold, with the consequence that there is the possibility of the wave action densities all having the same sign, and this in turn leads to the possibility of an explosive interaction.

Next, if the coefficients β_n $(n = p, q, r)$ have opposite signs (i.e. either two have positive values and one has a negative value, or vice versa), then it is a consequence of (3.8) that the interaction is contained and all three waves remain bounded in amplitude as the resonance surface is traversed. Some examples of typical interactions are shown in figure 1. In each case as τ increases from $-\infty$ to ∞, the amplitude $|A_r|$ of each wave oscillates with a slowly increasing amplitude before rapidly jumping to a new level and then oscillating with a slowly decreasing amplitude about this new level. Further details of the behaviour of the amplitudes during the interaction are given by Grimshaw (1987).

4. Wave interactions near a critical level

The analysis of the previous sections, and particularly §3, case (iii), has identified a scenario in which wave triads interact in the vicinity of certain resonance surfaces. We can envisage a sequence of such interactions in each of which a pair of waves interact to produce a third member of the resonant triad, determined by the resonance conditions (3.4*a*, *b*). During each such interaction wave action is exchanged and the amplitudes of each wave before and after the interaction are determined by solving equation (3.7*a*). To illustrate this process we choose the particular example of waves approaching a critical level. This choice is motivated by the series of papers by Brown & Stewartson (1980, 1982*a*, *b*) on the nonlinear processes affecting internal gravity waves near a critical level. They showed that wave reflection and transmission was determined by a hierarchy of wave interactions although their study differs in a crucial way from that described here in that the higher harmonics of the main incoming wave play a major role in the interaction process. Here we consider only the interactions of the dominant first harmonics of free waves, and further, for simplicity, will consider only the resonance conditions (3.4*a*, *b*). The question as to how much wave action is exchanged during the interaction will not be discussed here, and the reader is referred to Grimshaw (1987) for a detailed discussion of this aspect.

To model waves near a critical level, placed at $Z = 0$, we choose the Brunt–Väisälä frequency N to be constant, and put $\boldsymbol{u}_0 = (Z, 0, 0)$. The small parameter ϵ can be interpreted as a measure of $(Ri)^{-\frac{1}{2}}$ where Ri is the Richardson number of the basic flow at the critical level. We can assume that $\boldsymbol{\kappa}_{\mathrm{H}}$, the horizontal component of $\boldsymbol{\kappa}$, is a constant and parallel to $\boldsymbol{u}_0$, and to simplify the analysis we shall also assume that $m^2 \gg \kappa_{\mathrm{H}}^2$, where m is the vertical wavenumber component. This is an appropriate assumption for waves near a critical level. Then the dispersion relation (2.14) has the approximate solutions.

$$\hat{\omega} = \omega - kZ = \mp\frac{N|k|}{m}, \tag{4.1}$$

where $\boldsymbol{\kappa} = (k, 0, m)$ and we note that k is a constant. The alternate signs refer to waves whose vertical group velocity is positive (negative). Equation (4.1) is a partial differential equation for the phase Θ where we recall from (2.12) that $\omega = -\Theta_T$ and $m = \Theta_z$. To solve this equation we put $k = nK$ where n is an integer, $K > 0$, and define

$$\eta = \frac{KZT}{N}. \tag{4.2}$$

Then we put

$$\Theta = sN \ln T + kX + Nf(\eta), \tag{4.3}$$

where s is also an integer. Hence

$$\omega = -\frac{N}{T}(s + \eta f'(\eta)), \quad m = KTf'(\eta), \tag{4.4a, b}$$

and substituting these expressions into (4.1) we find that

$$\eta f'^2 + (n\eta + s) f' \mp |n| = 0. \tag{4.5}$$

The particular choice (4.3) is motivated by the work of Brown & Stewartson (1982a, b) who found waves of this form in their study of nonlinear waves near a critical level. The introduction of the integers n and s is to allow us to form a large family of waves so that, with particular choices of n and s, the resonance conditions (3.4a, b) can be satisfied. Although it would be tempting to identify the integer pair (n, s) as a wave harmonic we emphasize that here each integer pair corresponds to the dominant harmonic of a free wave. For each value of the integer pair (n, s) there are two solutions of (4.5) with positive group velocity, and two solutions with negative group velocity.

First we note that when $s = \operatorname{sign} n$, the two solutions of (4.5) with negative group velocity are

$$f' = -(\operatorname{sign} n)\,\eta^{-1}, \quad -n. \tag{4.6}$$

The corresponding frequency and vertical wavenumber components are

$$\omega = 0, \quad kZ - \frac{N}{T}\operatorname{sign} n, \tag{4.7a}$$

and

$$m = -\frac{N}{Z}\operatorname{sign} n, \quad -kT. \tag{4.7b}$$

The first of these solutions corresponds to a steady wave propagating downwards, and is just the steady wave analysed by Booker & Bretherton (1967), and found by them to undergo critical-layer absorption in the linear theory. Its trajectory is found by integrating the equation

$$\frac{\mathrm{d}Z}{\mathrm{d}T} = W = \pm\frac{N|k|}{m^2}, \tag{4.8}$$

where W is the vertical group velocity, and here we choose the lower sign since the wave has negative group velocity. Hence its trajectory is

$$\frac{KZ}{N} = (|n|\,T + \text{constant})^{-1}. \tag{4.9}$$

Thus in $Z > 0$, this represents a wave propagating towards the critical level at $Z = 0$ as $T \to \infty$, while in $Z < 0$ it represents a wave which propagates away from the critical level and $Z \to -\infty$ in a finite time. If it is assumed that the wave is generated for $T > 0$ far above the critical level, then its wavefront is given by $|n|\,\eta = 1$ and the wave occupies the region $|n|\,\eta \geqslant 1$. In the linear theory the wave is absorbed at the critical level. The second solution in (4.6) corresponds to critical-layer noise in the terminology of Brown & Stewartson (1980, 1982*a*, *b*), and represents a transient associated with the start-up process for the steady wave. Its trajectory is

$$\frac{KZ}{N} = \frac{1}{|n|\,T} + \text{constant}. \tag{4.10}$$

Thus this represents a wave which propagates down to some finite level of Z as $T \to \infty$.

Away from resonances (3.4*a*, *b*) the amplitudes of these waves can be determined by solving the equation for wave action (i.e. (2.22) with the right-hand side replaced by zero). For the boundary condition used by Brown & Stewartson (1982*a*) at the level where the waves are generated, we find that the amplitudes are proportional to $Z^{-\frac{1}{2}}$, and $T^{\frac{1}{2}}(|n|\,\eta-1)^{-1}$, respectively. The steady wave becomes infinite at the critical layer, while the critical-layer noise is singular at $|n|\,\eta = 1$. This latter result is a consequence of the asymptotic approximations inherent in a modulated wave theory, and the singularity is replaced by a boundary-layer structure in the full wave theory (Brown & Stewartson 1982*a*).

It will be shown below that the resonance surfaces are the level surfaces of η, and hence we may put $R = (\eta + \text{constant})$. It follows that, for each wave, δ (3.6*b*) is given by

$$\delta = \frac{K}{N}(Z+WT), \quad \text{or} \quad \delta = \frac{1}{T}(\eta \pm |n|\,(f')^2). \tag{4.11\,a, b}$$

To derive the second of these expressions we have used (4.4*b*) and (4.8), and the alternate signs again refer to a wave with positive (negative) vertical group velocity. Here we find that

$$\delta = \frac{\eta}{T}(1-|n|\,\eta), \quad \frac{1}{|n|\,T}(|n|\,\eta-1), \tag{4.12}$$

where these expressions refer to the steady wave, or the critical-layer noise respectively. It follows that the two waves approach a level surface of η from opposite sides if $\eta > 0$ and from the same side if $\eta < 0$. Also we recall that β (3.7*b*) is equal to sign $(\hat{\omega}\delta)$, and hence is given by

$$\beta = \operatorname{sign}\{n(|n|\,\eta-1)\}, \quad -\operatorname{sign}\{n(|n|\,\eta-1)\}, \tag{4.13}$$

for the steady wave, or the critical-layer noise, respectively.

Next we note that when $s = -\operatorname{sign} n$, the two solutions of (4.5) with positive group velocity are

$$f' = (\operatorname{sign} n)\,\eta^{-1}, \quad -n. \tag{4.14}$$

The corresponding frequency and vertical wavenumber components are

$$\omega = 0, \quad kZ + \frac{N}{T}\operatorname{sign} n, \tag{4.15a}$$

and

$$m = \frac{N}{Z}\operatorname{sign} n, \quad -kT. \tag{4.15b}$$

The first of these solutions corresponds to a steady wave propagating upwards, and the second solution again corresponds to critical-layer noise. Their trajectories and other properties are analogous to those described above for the corresponding waves with negative group velocity. Thus, for instance, the steady wave in $Z < 0$ represents a wave propagating towards the critical level as $T \to \infty$, while in $Z > 0$ it represents a wave which propagates away from the critical layer and $Z \to \infty$, in a finite time. On the other hand, the critical-level noise represents a wave which propagates up to some finite level of Z as $T \to \infty$.

Now we turn to the general case and seek solutions of (4.5) with positive group velocity. These are given by

$$2\eta f' = -(n\eta + s) \pm \{(n\eta+s)^2 + 4\eta|n|\}^{\frac{1}{2}}. \tag{4.16}$$

For $s(\operatorname{sign} n) < -1$, both branches are defined for all η, except possibly $\eta = 0$. For $s > 0$ the first branch is regular at $\eta = 0$ ($f' \approx |n|\, s^{-1}$), while the second branch is singular ($f' \approx -s\eta^{-1}$); for $s < 0$, the first branch is singular at $\eta = 0$ ($f' \approx -s\eta^{-1}$), while the second branch is regular ($f' \approx |n|\, s^{-1}$). As $\eta \to \infty$, $f' \sim \eta^{-1}$ for the first branch, and $f' \sim -n-(s+1)\,\eta^{-1}$ for the second branch, when $n > 0$; when $n < 0$, $f' \sim -n-(s-1)\,\eta^{-1}$ for the first branch, and $f' \sim -\eta^{-1}$ for the second branch. Comparing these expressions with (4.14), we can interpret the branch for which $f' \sim (\operatorname{sign} n)\,\eta^{-1}$ as $\eta \to \infty$ as a steady wave as $\eta \to \infty$, while the branch for which $f' \sim -n$ is interpreted as critical-layer noise. Similar considerations apply as $\eta \to -\infty$. The wave trajectories are found from (4.8), which using (4.2), (4.5) and (4.11*b*), can be written in the form

$$\frac{d\eta}{dT} = \delta = \frac{1}{T}(\eta + |n|\,(f')^{-2}). \tag{4.17a}$$

and also

$$\delta T f' = \pm\{(n\eta+s)^2 + 4\eta|n|\}^{\frac{1}{2}}. \tag{4.17b}$$

Integrating we find that the wave trajectories are given by

$$T(f'+n) = \text{constant}. \tag{4.18}$$

Depending on the sign of the constant, and $\operatorname{sign} n$, it can now be shown that each branch corresponds either to a wave which propagates from a finite value of Z as $T \to -\infty$ to $Z \to \infty$ in finite time, or to a wave which propagates from $Z \to -\infty$ at some finite time to a finite value of Z as $T \to \infty$. Both branches are hyperbola in the (Z, T)-plane, and some typical trajectories are shown in figure 2(*a*).

For $s(\operatorname{sign} n) \geqslant 0$, both branches are defined only for $\eta \geqslant \eta_1$ and $\eta \leqslant \eta_2$, where

$$|n|\,\eta_{1,2} = -\{2 + s(\operatorname{sign} n)\} \pm 2\{s(\operatorname{sign} n) + 1\}^{\frac{1}{2}}. \tag{4.19}$$

The two branches are equal at the turning points $\eta_{1,2}$ and we can regard the two branches as forming a single wave, one defined for $\eta \geqslant \eta_1$ and the other for $\eta \leqslant \eta_2$. Note that $\eta_2 < \eta_1 \leqslant 0$, and $\eta_1 = 0$ only for $s = 0$. The behaviour of the branches as $\eta \to 0$, or $|\eta| \to \infty$ is the same as that described in the previous paragraph. The wave trajectories are again given by (4.17*a*) and (4.18). In particular we note that since $\{(n\eta+s)^2+4\eta|n|\}^{\frac{1}{2}}$ vanishes at the turning points $\eta_{1,2}$, δ vanishes at the turning points which mark a transition between branches. Depending on the sign of the constant in (4.18), and sign n, it can now be shown that the wave trajectories correspond either to a wave which propagates from a finite value of Z as $T \to -\infty$ to a turning point (where there is an exchange of branches) and then to $Z \to \infty$ in finite time, or to a wave which propagates from $Z \to -\infty$ at some finite time to a turning point and then to a finite value of Z as $T \to \infty$. Again both branches are hyperbola in the (Z, T)-plane, and some typical trajectories are shown in figure 2(*b*).

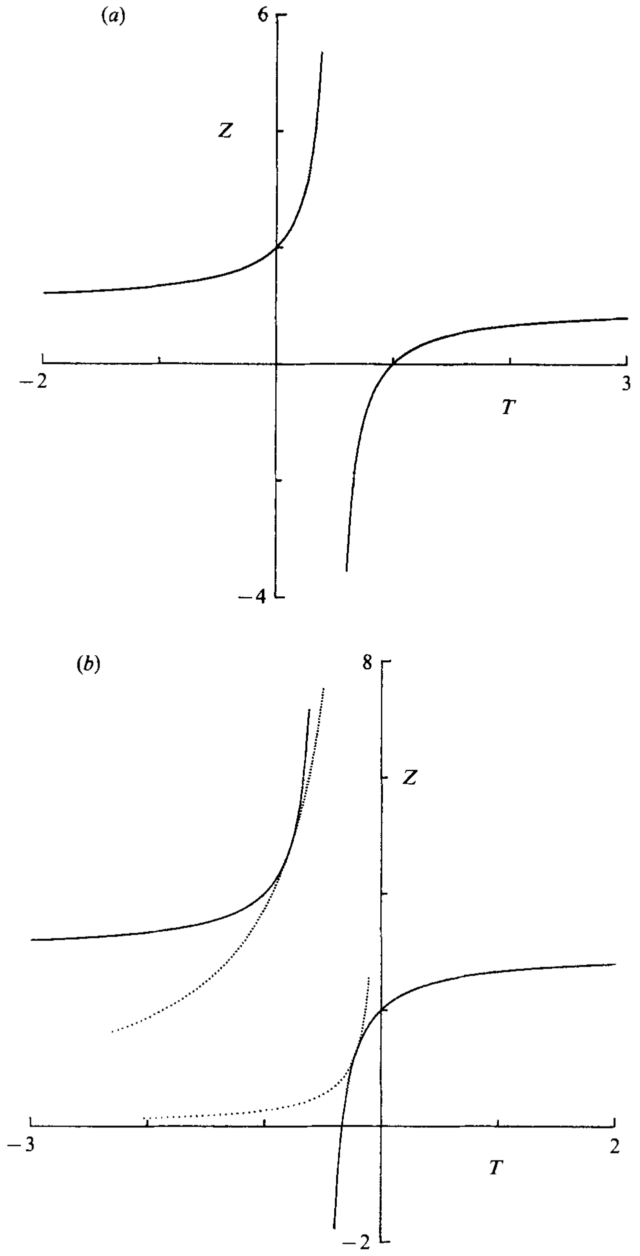

FIGURE 2. Typical trajectories for waves whose phases are given by (4.3) and (4.16). (a) $n = 2, s = -2$; (b) $n = 2, s = 2$, ..., denote turning points.

The solutions of (4.5) with negative group velocity are given by

$$2\eta f' = -(n\eta + s) \pm \{(n\eta + s)^2 - 4\eta|n|\}^{\frac{1}{2}}. \tag{4.20}$$

This can be analysed in a similar way to that described above for (4.16). We shall not give details except to observe that the transformation $\eta \to -\eta, n \to n, s \to -s$ and an exchange of branches takes (4.20) into (4.16).

With these preliminaries we now turn to an examination of the existence of resonant wave triads. Each wave can be described by $f'_{\pm}(\eta; n, s, \pm)$ where the subscript $\pm$ refers to the choice of branch, and the argument $\pm$ refers to the sign of the group velocity. The three members of the wave triad will be denoted by $(1,2,3)$ rather than (p,q,r), as used in §3. Then to satisfy the resonance conditions (3.4*a*, *b*) we find from (4.3) and (4.4*a*, *b*) that

$$n_1+n_2+n_3 = 0, \quad s_1+s_2+s_3 = 0, \tag{4.21 a, b}$$

and

$$\sum_{i=1}^{3} f'_{\pm}(\eta; n_i, s_i, \pm) = 0. \tag{4.21 c}$$

For a given pair of waves indexed by 1, 2 respectively, equations (4.21*a*, *b*) can be regarded as determining n_3 and s_3 and then (4.21*c*) determines one, or more, isolated values of η, which thus define the resonance surfaces. Effectively for a given input of waves 1, 2 equations (4.21*a*, *b*, *c*) determine a resonance site, and the third member of the wave triad, indexed by 3. From (4.16) and (4.20) it follows that (4.21*c*) can be written in the form

$$\sum_{i=1}^{3} \pm\{(n_i\eta+s_i)^2 \pm 4\eta|n_i|\}^{\frac{1}{2}} = 0, \tag{4.22}$$

where the first set of $(\pm)$ refers to the branch, and the second set of $(\pm)$ refers to the sign of the group velocity. Rejecting the possible solution $\eta = 0$, it can be shown that (4.22) reduces to a quadratic equation for η, and when this has real solutions, there are two possible resonance sites for each interaction. To be an actual resonance site, the allowed values of η must lie in the propagating zone for each wave (i.e. lie outside the region between the turning points $\eta_{1,2}$), and also must be consistent with the sign of the branches in (4.22). We also note here that, for each wave, it may be shown that

$$\beta = -\operatorname{sign}(WT)\operatorname{sign}(\delta f'T), \tag{4.23}$$

where β is defined by (3.7*b*), and the relative signs of β_1, β_2 and β_3 determine the nature of the interaction at each resonance site. For the case when β_i $(i = 1,2,3)$ all have the same sign it follows from (4.17*b*) and (4.22) that W_i $(i = 1,2,3)$ must have opposite signs. On the other hand, if β_i $(i = 1,2,3)$ have opposite signs it is possible to have an interaction in which W_i $(i = 1,2,3)$ all have the same sign.

In general there are many possible solutions of (4.21*a*, *b*, *c*). For simplicity we shall consider only the case of most interest when waves 1, 2 correspond to the special cases of a steady wave and critical layer noise, using the terminology of Brown & Stewartson (1980, 1982*a*, *b*). We shall also suppose that both these waves have negative group velocities. Thus, from (4.6) we put

$$f'(\eta; n_1, s_1, -) = -(\operatorname{sign} n_1)\,\eta^{-1}, s_1 = \operatorname{sign} n_1, \tag{4.24 a}$$

and

$$f'(\eta; n_2, s_2, -) = -n_2, \quad s_2 = \operatorname{sign} n_2. \tag{4.24 b}$$

Note that for these special cases the issue of the choice of branch is not relevant. With n_3, s_3 now determined from (4.21*a*, *b*), we use (4.21*c*) to solve for the possible resonance sites η, and also to determine wave 3. First, let us suppose that $n_1 n_2 > 0$, and hence $s_3 = -2\operatorname{sign} n_1$. Then wave 3 is described by $f'_{\pm}(\eta; n_3, s_3, \pm)$, and the resonance site is determined by (4.21*c*). We find that

$$n_1 n_2 \eta^2 + 2|n_1+n_2|\,\eta + 1 = 0 \quad \text{for } W_3 > 0, \tag{4.25 a}$$

or

$$n_1 n_2 \eta^2 + 1 = 0 \quad \text{for } W_3 < 0, \tag{4.25 b}$$

where W_3 is the vertical group velocity for wave 3. The solutions of (4.25*a*) are $\eta = -|n_1|^{-1}$, and $\eta = -|n_2|^{-1}$, and for $n_1 \gtrless n_2$, we must choose the $\pm$ branch of wave 3. The smaller of these two resonances in absolute value lies between the turning points η_1, η_2 of wave 3 and hence must be disregarded. The other will lie in the region $\eta < \eta_2$, which is a propagating zone for wave 3, provided that $\max(|n_1|, |n_2|) > (3+2\sqrt{3}) \min(|n_1|, |n_2|)$. However, since the non-propagating zone for wave 3 is $\eta_2 < \eta < \eta_1$, and $\eta_2 < \eta_1 \leqslant 0$, it follows that this resonance can only be realized if wave 1 is also defined in $\eta < 0$, rather than in $\eta > 0$. If wave 1 is defined in $\eta < 0$, then it represents a wave which is propagating away from the critical level at $Z = 0$. Hence this resonance is not of great practical interest. There are no real solutions of (4.25*b*). Thus, in summary, there are no realizable resonances of practical interest in this first case.

Second, we suppose that $n_1 n_2 < 0$ and hence $s_3 = 0$. The resonances are now given by

$$n_1 n_2 \eta^2 + 2 \max(|n_1|, |n_2|)\, \eta - 1 = 0 \quad \text{for } W_3 > 0, \tag{4.26a}$$

$$n_1 n_2 \eta^2 + 2 \min(|n_1|, |n_2|)\, \eta - 1 = 0 \quad \text{for } W_3 < 0. \tag{4.26b}$$

The solutions of (4.26*a*) are

$$\min(|n_1|, |n_2|)\, \eta = 1 \mp \left\{1 - \frac{\min(|n_1|, |n_2|)}{\max(|n_1|, |n_2|)}\right\}^{\frac{1}{2}}, \tag{4.27}$$

where the upper (lower) resonance corresponds to the $\pm$ branch of wave 3 if $n_1 > 0$, and vice versa if < 0. Since the turning points for wave 3 are $\eta_1 = 0$ and $\eta_2 = -4|n_3|^{-1}$, both of the resonances, being positive, lie in a propagating zone for wave 3. Thus both resonances can be realized in $\eta > 0$, and describe the interaction of a steady wave, and critical-layer noise, both propagating down to the critical level at $Z = 0$, to produce a third wave with positive group velocity. Depending on the choice of branch, that is, on the resonance which produces the wave, this third wave behaves as a steady wave, or as critical-layer noise, as $\eta \to \infty$. Now it can be shown that for the smaller resonance $\eta < \min(|n_1|^{-1}, |n_2|^{-1})$ and for the larger resonance $\eta > \max(|n_1|^{-1}, |n_2|^{-1})$. It then follows from (4.12) that at the smaller resonance $\delta_1 > 0$ and $\delta_2 < 0$, and at the larger resonance $\delta_1 < 0$ and $\delta_2 > 0$; in both cases the waves approach the resonance site from opposite sides. Also it follows from (4.17*b*) that $\delta_3(\operatorname{sign} n_1) > 0$ at both resonances. Further we note from (4.13) that $\beta_1 \beta_2 < 0$ so that the interaction of the three waves is always contained. Throughout this discussion the special case $|n_1| = |n_2|$ is excluded, as then $n_3 = 0$. Finally there are no real solutions of (4.26*b*).

5. Discussion

In this paper we have identified two kinds of resonant wave interactions. The first case, which we called (i) (strong resonance) in §3, occurs when the resonance conditions can be met globally. This is the case which is usually discussed in the literature. The second case, which we called (ii) (weak resonance) in §3, occurs when the resonance conditions can only be met locally, on certain resonance surfaces. Our main interest is in this second case, and while there are likely to be many scenarios where case (ii) could arise we have focused our attention on the situation when internal gravity waves interact near a critical level. Even then, with the family of waves restricted to those whose phases have the form (4.3), we find that there are many possible resonances. We may conclude that a critical layer is the site for vigorous wave–wave interactions.

The one scenario that we have explored in detail corresponds to the case when a steady wave and critical-layer noise propagating towards the critical level, interact there to produce a third wave which propagates back from the critical level and far away resembles either a steady wave or critical-layer noise. Thus the resonant interaction process can be said to have produced a reflected wave. Brown & Stewartson (1980, 1982*a*, *b*) have shown that wave forcing far from the critical level produces a steady wave propagating towards the critical level, and various transients. Near the critical level the dominant component in the transient wave field is critical-layer noise, being that component whose group velocity approaches zero near the critical level. They then showed that the driven second harmonic of the steady wave interacts with the critical-layer noise to produce a reflected wave, which resembles a steady wave far from the critical level. The mechanism described in this paper is similar, but has the crucial difference that only the first harmonics of each wave field are involved in the interaction.

Of course the mechanism described here is potentially the stronger, but it requires that the steady wave and the critical-layer noise correspond to different harmonics (i.e. $|n_1| \neq |n_2|$), whereas the Brown & Stewartson (1980, 1982*a*, *b*) mechanism allows the steady wave and the critical-layer noise to have the same harmonic behaviour (i.e. $|n_1| = |n_2|$) and hence can be driven by the same single harmonic source. To model the scenario described by Brown & Stewartson (1980, 1982*a*, *b*) within the present framework we choose $n_1 = 2$ for the steady wave, and $n_2 = -1$ for the critical-layer noise, in (4.24*a*, *b*). Thus the horizontal wavenumber component of the steady wave is exactly twice that for the critical-layer noise. However, note that $s_1 = 1$ and $s_2 = -1$. We then find that $n_3 = -1$ and $s_3 = 0$, and the phase is given by (4.3) where

$$2\eta f_3' = \eta \pm \{\eta^2 + 4\eta\}^{\frac{1}{2}}. \tag{5.1}$$

The resonance sites (4.27) are $\eta = \frac{1}{2}(2 \pm \sqrt{2})$. The reflected wave 3 generated at the upper resonance corresponds to the second $(-)$ branch in (5.1) which behaves as a steady wave far from the critical level, while that generated at the lower resonance site corresponds to the first $(+)$ branch in (5.1) which behaves as critical-layer noise. Significantly, since the upper resonance lies in $\eta > 1$, and the incoming steady wave is only defined in $\eta > 1$ with a wavefront at $\eta = 1$, only the resonant interaction producing the reflected steady wave can be realized.

In contrast the Brown & Stewartson mechanism considers the interaction of a second harmonic of an incoming steady wave ($n_1 = 1$ and $s_1 = 1$ in (4.22*a*)) with incoming critical layer noise ($n_2 = -1$ and $s_2 = -1$ in (4.22*b*)). Thus the resonance conditions (3.4*a*, *b*) are altered to

$$2\omega_1 + \omega_2 + \omega_3 = 0, \quad 2\boldsymbol{\kappa}_1 + \boldsymbol{\kappa}_2 + \boldsymbol{\kappa}_3 = 0. \tag{5.2a, b}$$

We now find that for the reflected outgoing wave $n_3 = -1$ and $s_3 = -1$, and the phase is given by (4.3) where

$$2\eta f_3' = \eta + 1 \pm \{\eta^2 + 6\eta + 1\}^{\frac{1}{2}}. \tag{5.3}$$

The resonance sites are $\eta = \frac{1}{2}(3 \pm \sqrt{5})$. The upper resonance site corresponds to the second $(-)$ branch in (5.3), which behaves as a steady wave as $\eta \rightarrow \infty$, while the lower resonance site corresponds to the first $(+)$ branch in (5.3), which behaves as critical layer noise as $\eta \rightarrow \infty$. Again the upper resonance site lies in $\eta > 1$, and since the incoming steady wave is only defined in $\eta > 1$, only the resonant interaction producing the steady wave can be realized. These results agree with those of Brown & Stewartson (1980, 1982*a*, *b*) who also calculated the amplitude of the reflected wave. Apart from deriving the equations (3.7*a*, *b*) for the determination of the amplitudes during the interaction, we have not discussed the amplitudes in this paper. The details of this calculation are complex and are the subject of current work.

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